An Elastic-Plastic Formulation for a Cylindrical Shell Finite Element

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Theme

AN elastic-plastic formulation is presented for a curved rectangular thin shell finite element. Elastic-perfectly plastic material behavior is assumed. The von Mises yield criterion is used to define the limit of plasticity. Plastic strain distributions and a distribution for an elastic-plastic boundary in the element are defined. Incremental stress-strain relations are developed for use in the plastic range. With this formulation and an incremental approach, yielding can be traced in a deforming shell and displacements, and stresses can be found at each increment of load.

Contents

The singly curved, rectangular, elastic shell finite element developed in Ref. 1 is chosen for this plastic analysis. The element includes the 6 rigid-body modes explicitly. The element has relatively fewer number of degrees-of-freedom (24) as compared with other shell elements. This permits one to employ relatively more elements to model the boundaries of the plastic regions.

The material is assumed as elastic-perfectly plastic. A distinct boundary is assumed to exist between the elastic and plastic regions. The elastic limit is defined by the von Mises yield criterion

$$\sigma_0^2 = J_2 = \sigma_\xi^2 + \sigma_\eta^2 - \sigma_\xi \sigma_\eta + 3\sigma_{\xi\eta}{}^2 \tag{1}$$

where σ_0 is the yield stress in tension.

The plastic (initial) strain distributions are assumed to vary bilinearly in the surface of the finite element and vary linearly from their values at the upper or lower surface to zero at the elastic-plastic boundary. These distributions are

$$\{\epsilon\} = [S_p]\ \{\epsilon_0\} \tag{2}$$

for the upper or lower plastic regions, respectively. The vector $\{\epsilon\}$ contains three plastic strain components ϵ_ξ, ϵ_η, and $\epsilon_{\xi\eta}$ and the vector $\{\epsilon_0\}$ contains the values of $\{\epsilon\}$ at either the four upper surface corners or the four lower surface corners.

The stresses in the shell in the elastic range are

$$\left\{\begin{aligned} \sigma_\xi &= \frac{E}{1-\nu^2}\,[(e_\xi + \nu e_\eta) + \zeta(k_\xi + \nu k_\eta)] \\ \sigma_\eta &= \frac{E}{1-\nu^2}\,[(e_\eta + \nu e_\xi) + \zeta(k_\eta \nu k_\xi)] \\ \sigma_{\xi\eta} &= \frac{E}{2(1+\nu)}\,(e_{\xi\eta} + \zeta k_{\xi\eta}) \end{aligned}\right. \tag{3}$$

where the middle surface strain components and the curvature terms are related to the displacement derivatives as

$$\left\{\begin{aligned} e_\xi &= \frac{\partial u}{\partial \xi} \\ e_\eta &= \frac{\partial v}{\partial \eta} + \frac{w}{r} \\ e_{\xi\eta} &= \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \end{aligned}\right. \qquad \left\{\begin{aligned} k_\xi &= -\frac{\partial^2 w}{\partial \xi^2} \\ k_\eta &= \frac{1}{r}\frac{\partial v}{\partial \eta} - \frac{\partial^2 w}{\partial \eta^2} \\ k_{\xi\eta} &= 2(\frac{1}{r}\frac{\partial v}{\partial \xi} - \frac{\partial^2 w}{\partial \xi \partial \eta} \end{aligned}\right. \tag{4}$$

and ζ is the distance measured from the middle surface of the shell. The particular value of ζ that defines the elastic-plastic boundary is designated as ρ. Substituting Eqs. (3) and (4) in (1) results in

$$A\rho^2 + 2B\rho + C - \sigma_0^2 = 0 \tag{5}$$

where

$$A = \frac{E^2}{(1+\nu)^2}\,[(k_\xi - k_\eta)^2 + \tfrac{3}{4}k_{\xi\eta}^2 + k_\xi k_\eta + \frac{\nu}{(1-\nu)^2}\,(k_\xi + k_\eta)^2]$$

$$B = -\frac{E^2}{(1-\nu^2)^2}\,\{e_\xi[(1-\nu+\nu^2)k_\xi - \tfrac{1}{2}(1-4\nu+\nu^2)k_\eta] + e_\eta[-\tfrac{1}{2}(1-4\nu+\nu^2)k_\xi + (1-\nu+\nu^2)k_\eta] + \tfrac{3}{4}(1-\nu)^2 e_{\xi\eta}k_{\xi\eta}\}$$

$$C = \frac{E^2}{(1+\nu)^2}\,[(e_\xi - e_\eta)^2 + \tfrac{3}{4}e_{\xi\eta}^2 + e_\xi e_\eta + \frac{\nu}{(1-\nu)^2}\,(e_\xi + e_\eta)^2] \tag{6}$$

The two roots of ρ in Eq. (5) defines the upper and lower plastic boundaries.

For elastic-perfectly plastic material behavior, the stress increment vector must remain tangent to the yield surface and the plastic strain increment vector must remain normal to the yield surface. These two conditions can be expressed by von Mises yield condition in an incremental form

$$(\sigma_\xi - \tfrac{1}{2}\sigma_\eta)\Delta\sigma_\xi + (\sigma_\eta - \tfrac{1}{2}\sigma_\xi)\Delta\sigma_\eta + 3\sigma_{\xi\eta}\Delta\sigma_{\xi\eta} = 0$$

$$\frac{\Delta\epsilon_\xi}{(\sigma_\xi - \tfrac{1}{2}\sigma_\eta)} = \frac{\Delta\epsilon_\eta}{(\sigma_\eta - \tfrac{1}{2}\sigma_\xi)} = \frac{\Delta\epsilon_{\xi\eta}}{3\sigma_{\xi\eta}} \tag{7}$$

In the presence of plastic strains, the total strain increments are composed of elastic and plastic strain increments

$$\{\Delta e^T\} = \{\Delta e^e\} + \{\Delta\epsilon\} \tag{8}$$

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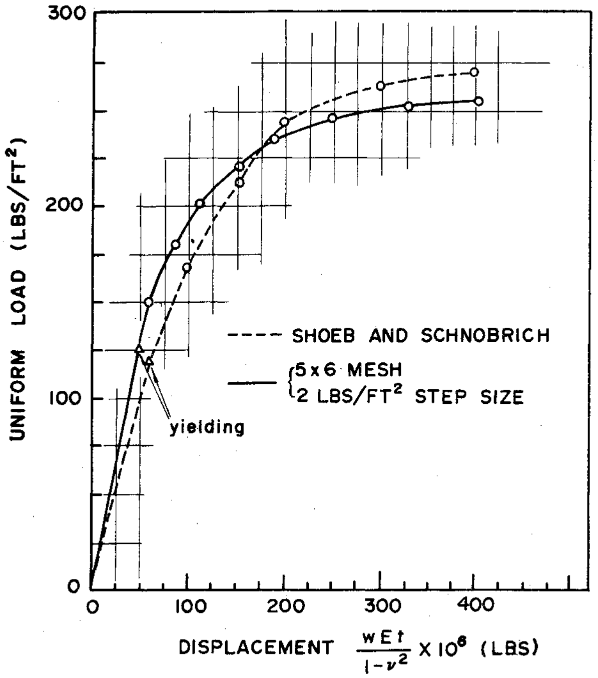

Fig. 1 Uniform gravity load vs the deflection w at the center of the free edge of the cylindrical panel.

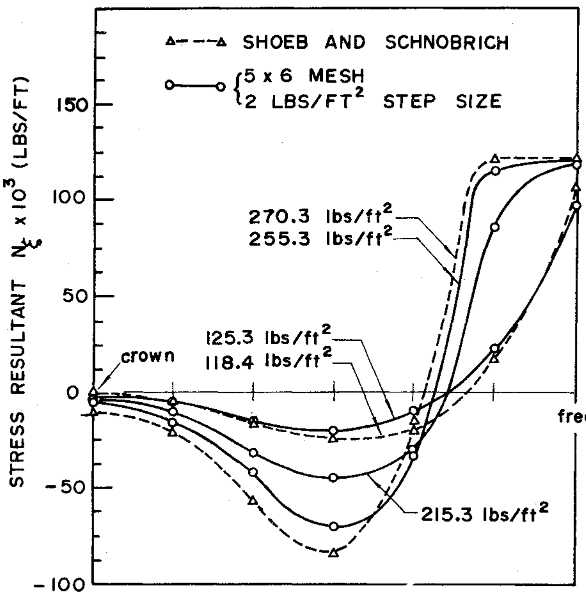

Fig. 2 Variation of stress resultant N_ξ along the midspan at various load levels.

Equations (7) and (8) provide 6 simultaneous equations. Once the total strain increments are known, the stress and plastic strain increments follow readily.

The elastic strain energy for an element can be written as

$$u = \tfrac{1}{2}\iiint_V \lfloor e^T \rfloor [E] \{e^T\} \mathrm{d}V - \iiint_{V_p} \lfloor e^T \rfloor [E] \{\epsilon\} \mathrm{d}V + \tfrac{1}{2}\iiint_{V_p} \lfloor \epsilon \rfloor [E] \{\epsilon\} \mathrm{d}V \tag{9}$$

where $[E]$ is the elastic stress-strain matrix, and V_p is the volume of the plastic portions in the element.

For the element given in Ref. 1, the total strain components are related to the 24 nodal degrees-of-freedom $\{d_0\}$ as

$$\{e^T\} = [P^*][T]^{-1}\{d_0\} \tag{10}$$

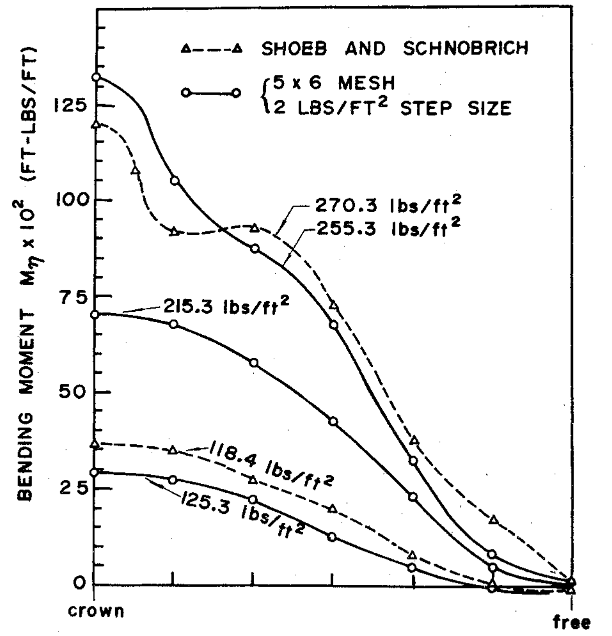

Fig. 3 Variation of bending moment M_η along the midspan at various load levels.

where $[T]$ related $\{d_0\}$ to the 24 coefficients $\{a\}$ in the displacement functions and $[P^*]$ relates the three strain and curvature components to $\{a\}$.

Substituting Eqs. (2) and (10) in (9) and using the principle of virtual work, the element stiffness formulation is obtained

$$\{p\} = [k]\{d_0\} - [k^*]\{\epsilon_0\} \tag{11}$$

where $\{p\}$ is the vector of 24 nodal forces, and the elastic and plastic stiffness matrices are

$$[k] = [T^{-1}]^T \iiint_V [P^*]^T [E][P^*] \mathrm{d}V [T^{-1}] \tag{12}$$

$$[k^*] = [T^{-1}]^T \iiint_{V_p} [P^*]^T [E][S_p] \mathrm{d}V$$

Equation (11) can be solved in an incremental form by contributing the plastic portion as fictitious forces at each increment. Due to the incremental nature of the procedure, the stress point associated with the developed plasticity moves along a tangent to the yield surface. The stress point is brought back to the yield surface if the movement away from the surface is beyond a specified tolerance.

A cylindrical panel with two straight longitudinal edges free and the two circular edges simply-supported under the lateral gravity load was analyzed. The thickness, length, radius, and half opening angle are assumed as 0.3125 ft, 62 ft, 31 ft, and 40°, respectively. The modulus of elasticity, Poisson's ratio, and yield stress are assumed as 4.32×10^8 lb/ft², 0, and 3.8×10^5 lb/ft², respectively. This example was analyzed previously in Ref. 2 by a deformable node-rigid bar model and a finite difference technique.

A convergence study was first performed and a 5×6 mesh for idealizing a quadrant of the shell with equal load step size of 2 psf were chosen. The deflection path at the center of the free edge is shown in Fig. 1. Results from Ref. 2 is also shown. The longitudinal stress resultant N_ξ and the circumferential bending moment M_η along the midspan are shown in Figs. 2 and 3, respectively. Near the free edge, N_ξ is seen to converge to the yield value.

References

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